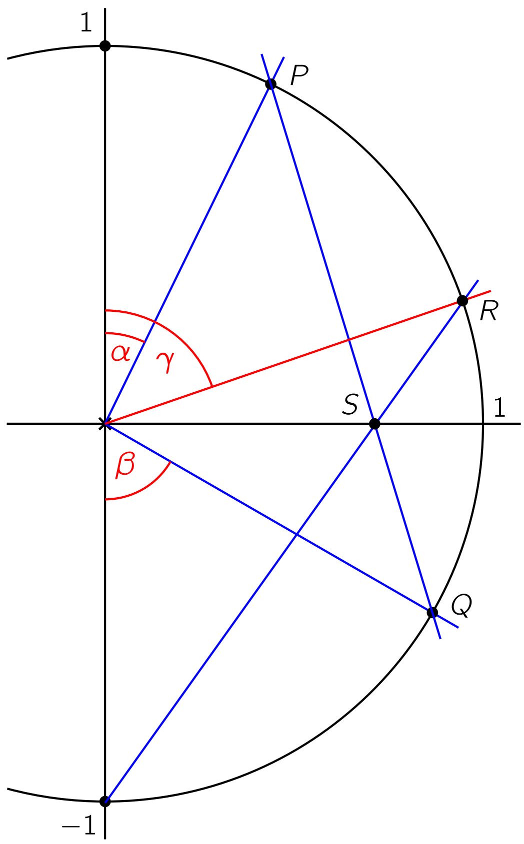
Velocity Addition in STR with Ruler and Compass

1. The Construction of $a \oplus b$ following Jerzy Kocik
2. An Implication concerning "Half of the Speed"
3. The Abelian Group of Velocities in STR
4. The Addition of Epstein-Angles following Hepp
5. The Addition of Epstein-Angles following Hepp and Gubler
6. The Epstein-Group of Acute Angles

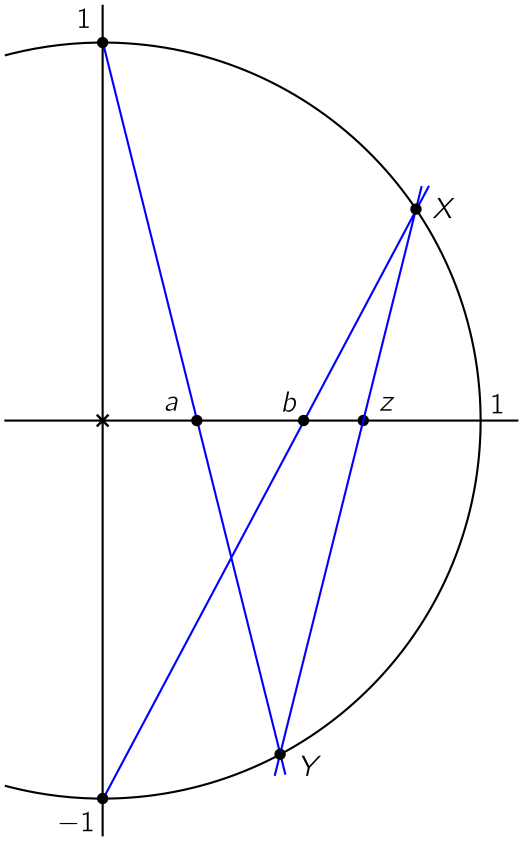


English version 1.01 , December 4th 2013

Alfred Hepp, Bergün and Martin Gubler, Frauenfeld

1 The Construction of $a \oplus b$ following Jerzy Kocik

Jerzy Kocik gives us in Am. J. Phys. Vol. 80-8, August 2012, p.737f the following construction for $a \oplus b$:



The velocities $a = \dfrac{v}{c}$ and $b = \dfrac{u}{c}$ given as dimensionless quantities in the interval [-1, 1]

- mark the position of a and b on the x-axis
- draw the chord through (0/1) and (a/0) and get Y
- draw the chord through (0/-1) and (b/0) and get X
- draw the chord through X and Y and get (z/0)

Then $z = a \oplus b = \dfrac{a+b}{1 + a \cdot b}$ holds.

Problems:

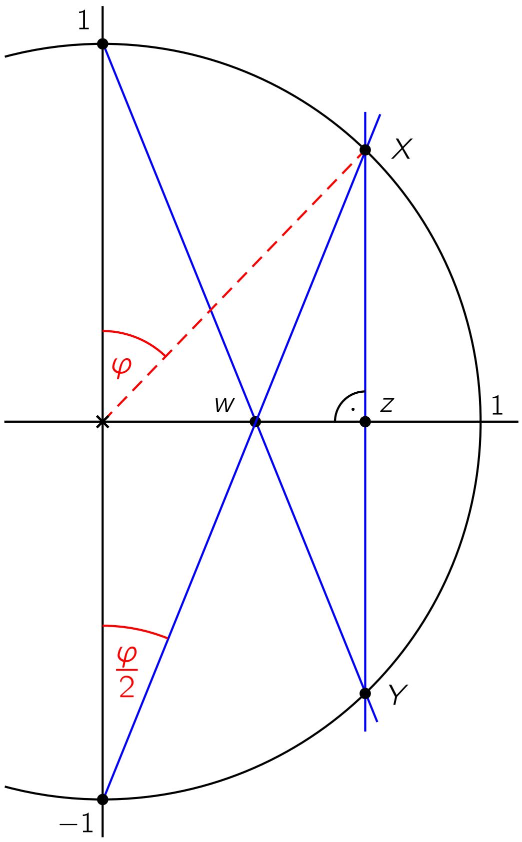
1. Construct $0.5 \oplus 0.6$.

2. Construct $a \oplus 1$, $1 \oplus 1$, $a \oplus (-1)$, $a \oplus (-a)$.

3. Calculate the coordinates of X and Y in general.

4. Use the results of **3.** and calculate the value of z. Thus give a proof of the correctness of Kocik's construction!

2 An Implication concerning "Half of the Speed"



Let $z \in]\text{-}1, 1[$ be given.

- the line through (z/0) perpendicular to the x-axis yields X and Y
- the chords from (0/-1) to X and from (0/1) to Y both yield w
- following section **1** we get $w \oplus w = z$

So w is "half of the speed" of z in STR, and

we have $$w = \frac{z}{1 + \sqrt{1 - z^2}} \qquad (1)$$

For the *central angle* φ we have $\sin(\varphi) = z$. So φ is the Epstein-angle of velocity z.

For the *inscribed angle* $\frac{\varphi}{2}$ we have $\tan(\frac{\varphi}{2}) = w$.

Problems:

1. Construct $0.5 \oplus 0.5$.
2. Construct "half of the speed" of $0.9 \cdot c$.
3. Use the definition of $\oplus$ and show (1).
4. The above figur shows: $z = \sin(\varphi)$ and $w = \tan(\frac{\varphi}{2})$. Use some goniometric transformations and show $\tan(\frac{\varphi}{2}) \oplus \tan(\frac{\varphi}{2}) = \sin(\varphi)$.
5. Derive equation (1) using the triangle X, (0/0) und (z/0): Why is the following equation correct?
$$\frac{z - w}{\sqrt{1 - z^2}} = \tan\left(\frac{\varphi}{2}\right) = w$$
6. The angles (0/0) X (w/0) and (w/0) X (z/0) are both equal to $\frac{\varphi}{2}$. What follows for the relation of $(z - w)$ to w using one of Apollonius' theorems? This gives another simple proof of (1).

3 The Abelian Group of Velocities in STR

The operation

$$a \oplus b = \frac{a+b}{1+a \cdot b}$$

turns the open interval $]\text{-}1, 1[$ into an Abelian group:

i) the operation is obviously commutative

ii) zero is the additive identity element

iii) $-a$ is the additive inverse number of a

iv) the associativity law holds (see problem 1)

We need this to proof the following

Lemma: Let $p \oplus p = a$, $q \oplus q = b$ and $w \oplus w = z$. Then the following statements are equivalent:

i) $a \oplus b = z$

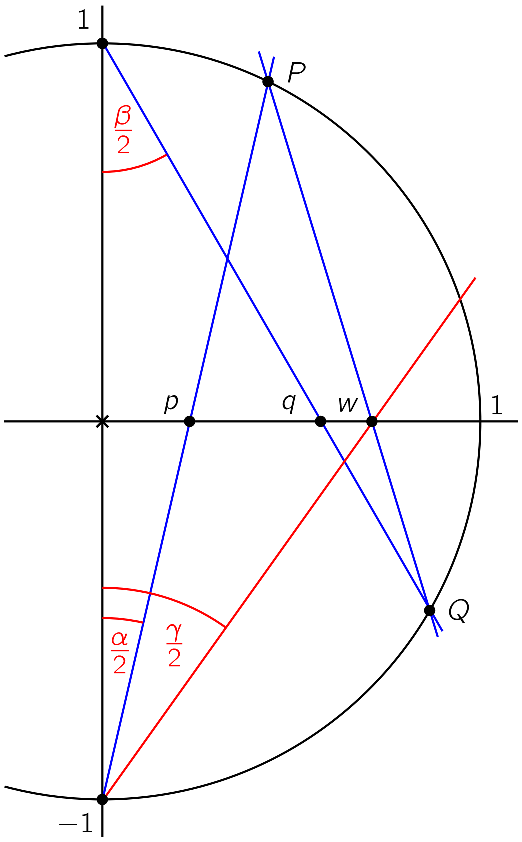
ii) $p \oplus q = w$

In other words: Half of the speed of the sum equals the sum of the half-speeds ...

Problems:

1. Give a proof of the associativity of the operation $\oplus$.
2. Give a proof of the Lemma.
3. Why is it important to exclude the numbers 1 and -1? In section **1** we did construct the additions $a \oplus 1$ and $a \oplus (-1)$, the result is well defined?

4 The Addition of Epstein-Angles following Hepp



Let the velocities a and b be given by their Epstein-angle:

$a = \sin(\alpha)$ and $b = \sin(\beta)$.

- draw the inscribed angles $\frac{\alpha}{2}$ and $\frac{\beta}{2}$ with vertexes in (0/-1) and (0/1) and get P and Q as well as p und q
- draw the chord PQ and get w and $\frac{\gamma}{2}$

then $\gamma = 2 \cdot \frac{\gamma}{2}$ is the Epstein-angle corresponding to $a \oplus b$

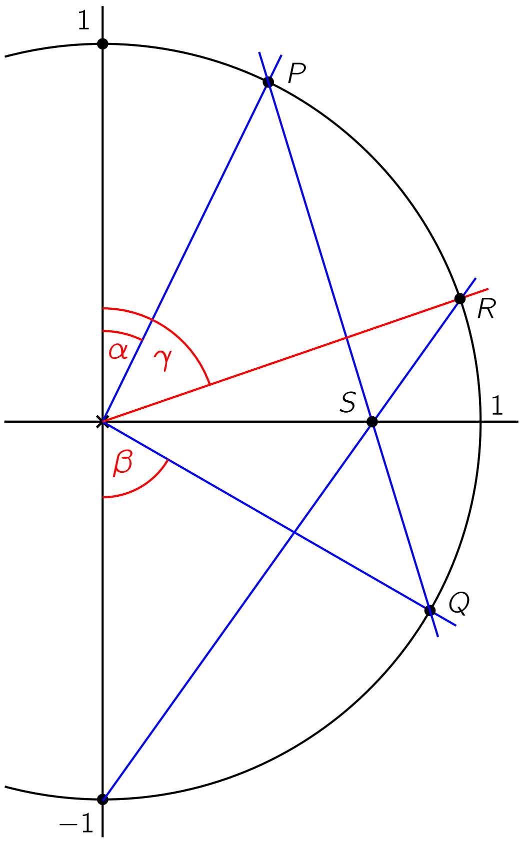
Proof:

- section **2** shows that $p \oplus p = a$ and $q \oplus q = b$
- section **1** shows that $w = p \oplus q$
- section **3** shows that $w \oplus w = a \oplus b$
- section **2** shows that $\frac{\gamma}{2}$ is half of the Epstein-angle of $a \oplus b$

The solution of problem **3** in section **1** is $P = (a/\sqrt{1-a^2}\,)$ and $Q = (b/-\sqrt{1-b^2}\,)$. Calculating the intersection of the straight line PQ with the x-axis you get a term expressing w. You could show then by direct, but tedious calculations that $w \oplus w = a \oplus b$ holds.

Never heard of Epstein-angles? Then read chapter C of "Epstein Explains Einstein" on http://www.relativity.li/en/epstein2/read/c0_en/.

5 The Addition of Epstein-Angles following Hepp and Gubler



Using central angles instead of inscribed angles we can avoid bisection and doubling of angles:

Let the velocities a and b again be given by their Epstein-angles α and β with $a = \sin(\alpha)$ and $b = \sin(\beta)$.

- draw the central angles α and β and get P and Q
- draw the chord PQ and get S
- the chord through (0/-1) and S yields R and γ

Proof:

P and Q have the same position as in section **4**, hence $S = (w/0)$.

Following **2** we have $w \oplus w = a \oplus b$ and $\sin(\gamma) = a \oplus b$.

6 The Epstein-Group of Acute Angles

The addition $\oplus$ on the open intervall $]\text{-}1, 1[$ is easily transferred to the set of acute angles in the open intervall $]\text{-}90°, 90°[$. The mapping

$$\varphi \longmapsto \sin(\varphi)$$

is one-to-one, and we define the addition $\oplus$ for acute angles by

$$\underbrace{\alpha \oplus \beta}_{\text{new, for angles}} := \sin^{-1}\left(\sin(\alpha) \underbrace{\oplus}_{\text{as before, for velocities } \frac{v}{c}} \sin(\beta)\right)$$

Both groups are isomorphic, thus we did not create anything new. However, we enjoy the fact that in **5** we could present a ruler-and-compass construction for this addition $\oplus$ of Epstein angles.